Derivation of $-1 \leq \rho(s,t) \leq 1$ from definition of covariance

Given covariance is,

$$\gamma(s,t) = \gamma(x_s, x_t) = E[(x_s - \mu_s)(x_t - \mu_t)]$$

and,

$$\gamma(s,s) = \gamma(x_s, x_s) = E[(x_s - \mu_s)^2] = Var(s)$$

and,

$$\rho(s,t) = \frac{\left(\gamma(s,t)\right)^2}{\gamma(s,s)\gamma(t,t)}$$

show ,

$$-1 \leq \rho(s,t) \leq 1$$

Deal with $\left(\gamma(s,t)\right)^2$ first,

$$\left(\gamma(s,t)\right)^2 = (E[(x_s - \mu_s)(x_t - \mu_t)])^2 = (E[(x_s x_t - \mu_s x_t - \mu_t x_s + \mu_s \mu_t)])^2$$

$$= (E[x_s x_t] - \mu_s E[x_t] - \mu_t E[x_s] + \mu_s \mu_t)^2 = (E[x_s x_t] - \mu_s \mu_t - \mu_t \mu_s + \mu_s \mu_t)^2$$

$$= (E[x_s x_t] - \mu_s \mu_t)^2$$

$$= (E[x_s x_t])^2 - 2E[x_s x_t]\mu_s \mu_t + {\mu_s}^2 {\mu_t}^2$$

Deal with $\gamma(s,s)\gamma(t,t)$ next,

$$\gamma(s,s)\gamma(t,t) = E[(x_s - \mu_s)^2]E[(x_t - \mu_t)^2] = E[({x_s}^2 - 2\mu_s x_s + {\mu_s}^2)]E[({x_t}^2 - 2\mu_t x_t + {\mu_t}^2)]$$

$$= E[{x_s}^2 - 2\mu_s x_s + {\mu_s}^2]E[{x_t}^2 - 2\mu_t x_t + {\mu_t}^2] = \left(E[{x_s}^2] - 2\mu_s E[x_s] + {\mu_s}^2\right)\left(E[{x_t}^2] - 2\mu_t E[x_t] + {\mu_t}^2\right)$$

$$= \left(E[{x_s}^2] - 2{\mu_s}^2 + {\mu_s}^2\right)\left(E[{x_t}^2] - 2{\mu_t}^2 + {\mu_t}^2\right) = (E[{x_s}^2] - {\mu_s}^2)(E[{x_t}^2] - {\mu_t}^2)$$

$$= E[{x_s}^2]E[{x_t}^2] - E[{x_s}^2]{\mu_t}^2 - E[{x_t}^2]{\mu_s}^2 + {\mu_s}^2{\mu_t}^2$$

Next substitute results into $\rho(s,t)$,

$$\rho(s,t) = \frac{(E[x_s x_t])^2 - 2E[x_s x_t]\mu_s \mu_t + {\mu_s}^2 {\mu_t}^2}{E[{x_s}^2]E[{x_t}^2] - E[{x_s}^2]{\mu_t}^2 - E[{x_t}^2]{\mu_s}^2 + {\mu_s}^2{\mu_t}^2}$$

When the variables are uncorrelated, the numerator becomes,

$$(E[x_s x_t])^2 - 2E[x_s x_t]\mu_s \mu_t + {\mu_s}^2 {\mu_t}^2 = (E[x_s]E[x_t])^2 - 2E[x_s]E[x_t]\mu_s \mu_t + {\mu_s}^2 {\mu_t}^2$$

$$= (\mu_s \mu_t)^2 - 2\mu_s \mu_t \mu_s \mu_t + {\mu_s}^2 {\mu_t}^2 = {\mu_s}^2 {\mu_t}^2 - 2{\mu_s}^2 {\mu_t}^2 + {\mu_s}^2 {\mu_t}^2 = 0$$

$$\therefore \rho(s,t) = 0$$

When variables are perfectly correlated, $x_t = a x_s$, where, a, is a constant such that, $\mu_t = E[x_t] = E[a x_s] = aE[x_s] = a\mu_s$, then

$$\rho(s,t) = \frac{(E[x_s x_t])^2 - 2E[x_s x_t]\mu_s \mu_t + {\mu_s}^2 {\mu_t}^2}{E[{x_s}^2]E[{x_t}^2] - E[{x_s}^2]{\mu_t}^2 - E[{x_t}^2]{\mu_s}^2 + {\mu_s}^2{\mu_t}^2} = \frac{(E[x_s a x_s])^2 - 2E[x_s a x_s]\mu_s a \mu_s + {\mu_s}^2 a^2 {\mu_s}^2}{E[{x_s}^2]E[a^2{x_s}^2] - E[{x_s}^2]a^2{\mu_s}^2 - E[a^2{x_s}^2]{\mu_s}^2 + {\mu_s}^2 a^2 {\mu_s}^2}$$

$$= \frac{(aE[{x_s}^2])^2 - 2aE[{x_s}^2]a{\mu_s}^2 + {\mu_s}^2 a^2 {\mu_s}^2}{a^2E[{x_s}^2]E[{x_s}^2] - E[{x_s}^2]a^2{\mu_s}^2 - E[a^2{x_s}^2]{\mu_s}^2 + {\mu_s}^2 a^2 {\mu_s}^2} = \frac{a^2(E[{x_s}^2])^2 - 2a^2E[{x_s}^2]{\mu_s}^2 + {\mu_s}^2 a^2 {\mu_s}^2}{a^2(E[{x_s}^2])^2 - a^2E[{x_s}^2]{\mu_s}^2 - a^2E[{x_s}^2]{\mu_s}^2 + {\mu_s}^2 a^2 {\mu_s}^2}$$

$$= \frac{a^2(E[{x_s}^2])^2 - 2a^2E[{x_s}^2]{\mu_s}^2 + {\mu_s}^2 a^2 {\mu_s}^2}{a^2(E[{x_s}^2])^2 - 2a^2E[{x_s}^2]{\mu_s}^2 + {\mu_s}^2 a^2 {\mu_s}^2} = 1$$

$$\therefore \rho(s,t) = 1$$